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Numerical Analysis of Radiation Heat Transfer Using the Lattice Boltzmann Method and the Finite Volume Method

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Abstract

This article reports the result of two popular numerical methods, viz., finite volume method (FVM) and lattice Boltzmann method (LBM) used to calculate the radiation heat transfer within an enclosure. 2-D rectangular enclosure with absorbing, emitting and scattering participating medium is considered. In terms of collision and streaming, the present approach of LBM for the radiative heat transfer is similar to those being used in fluid dynamics and heat transfer for the analysis of conduction and convection. However, in the present LBM approach in order to mitigate the effect of isotropy in the polar direction the number of lattices employed is greater than those being used for 2-D system. Distribution of heat flux and emissive power for different values of extinction coefficient has been obtained using both methods. The good agreement has been found between LBM and FVM results. The number of iteration and CPU times has also been compared. It is found that for the convergence of solution the LBM required a greater number of iterations as FVM but LBM computationally much faster than FVM.

Keywords: Finite Volume Method; Lattice Boltzmann Method; Radiation Heat Transfer

1. Introduction

Study of volumetric radiation is vital in many high temperature thermal appliances and processes[1]. The systems which necessitate a proper examination of thermal radiation include but not limited to the design of boilers, furnaces, internal combustion engines and insulations[2]. Analysis of phase change process of semitransparent materials like glass and semiconductor materials is another case where knowledge of the volumetric radiation is required [3-6].

The occurrence of absorption, emission and scattering characterize radiative transport in a participating medium as a volumetric phenomenon[7]. Radiation is different from other two modes of heat transfer,

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viz., conduction and convection as it not only depends upon spatial and temporal dimensions but also varies with changes in wavelength, polar and azimuthal angles. Owing to the angular dependence, the governing radiative transfer equation (RTE) comes out to be an integrodifferential one [7]. And again, it is the angular dependence which makes it difficult to calculate the radiative information in a combined mode heat transfer problem involving thermal radiation [8-9]

The finite volume method (FVM) is widely used to calculate the radiative information in radiative heat transfer problems [10-11]. Although this method is a variant of the DOM [12] yet it does not undergo the false-scattering unlike the DOM[10]. Chai reported that the ray-effect in this method is less pronounced [10]. As the concept employed by FVM for the radiative heat transfer and for the CFD is same, in conjugate mode problems, the FVM grids employed in the solution of the momentum and energy equations are compatible with its computational grids [13-14]. Consequently, although the FVM for the radiative heat transfer is a method introduced only 15 years ago, it is much more popular than the DTM, and the DOM. However, it is pertinent to note that cost of radiation is still quite high even with the FVM. This is why quest for a computationally more competent method is still not over.

During the past decade, the lattice Boltzmann method (LBM) has gained popularity as a substitute to the already existing computational fluid dynamics (CFD) solving techniques like the finite element method (FEM), the finite difference method (FDM) and the finite volume method (FVM) [15-19].

Nomenclature	
I - intensity	$_s$ -scattering coefficient

c - speed of light	-Stefan-Boltzmann constant
s - geometric distance	-relaxation time
e -velocity in the discrete direction *i*	-scattering albedo
I_b - blackbody intensity, $\frac{T^{4}}{}$	-polar angle
q -heat flux	-azimuthal angle
N -number of discrete directions	**Superscripts**
p - scattering phase function	*l* -index for the discrete direction
t -time	-non-dimensional quantities
U -speed	(*eq*) -equilibrium
w -weight	**Subscripts**
t_p -pulse width	*d* -diffuse
Greek Symbols	*c* -collimated
-extinction coefficient	*m* -modified
k_a -absorption coefficient	*e* , *w*, *s* , *n* -directions

The LBM has been applied to solve the energy equation of transient conduction and radiation heat transfer in a planar medium both in the case of heat generation and without it[20]. They used the discrete transfer method (DTM) to compute the radiative information[21]. Mishra *et al*. solved the energy equation of a transient conduction–radiation heat transfer in a 2-D square enclosure using the LBM method[22]. Their study comprises of the application of the collapsed dimension method (CDM) for computation of the radiative information[23]. Raj et al studied the solidification of a semitransparent planar layer was extended by applying the LBM[6]. He calculated the radiative information by using the DTM. Gupta et al. solved the energy equation of a temperature dependent transient conduction and radiation heat transfer in a planar medium by using the theory of inconsistent relaxation time in the LBM [24].

In all of the above-mentioned applications to the conduction–radiation heat transfer problems, the LBM was compared to other methods and was established as the most accurate one regarding the results. Apart from this, the LBM was applied for solution of energy equation whereas the DTM, the CDM and the DOM were used for the determination of radiative information and their compatibilities were found out.

Keeping the above discussion in view, the present work is aimed at comparing the results as well as the computational efficiencies of the LBM using more number of directions as proposed in [25] and the FVM for different classes of radiation heat transfer problems in a 2-D rectangular enclosure with absorbing, emitting and scattering medium. The results of the two methods are compared for different parameters and their computational efficiencies are reported.

Mathematical Formulation

The Radiative Transfer Equation (RTE) for a gray medium can be written as [26]:

$$\frac{dI(r,\hat{s},t)}{ds} = \hat{s}.\nabla I(r,\hat{s},t) = -\beta I(r,\hat{s},t) + k_a(r) I_b + \frac{\sigma_s}{4\pi}\int_{4\pi} I(r,\hat{s},t)\, p(\Omega,\Omega^{'})\, d\Omega^{'} \quad (1)$$

where the term on left hand side indicates the intensity gradient with respect to space. The first term on right hand side represents the attenuation of radiation intensity due to absorption whereas the second and third terms indicate the augmentation part due to emission from medium and in scattering from all other directions respectively. We note, furthermore, the following: $\beta = k_a + \sigma_s$ is the extinction coefficient; for isotropic scattering, $p(\Omega,\Omega') = 1$; the scattering albedo 'ω' is defined as $\omega = \sigma_s / k_a + \sigma_s$.

The value scattering albedo ω for a purely absorbing and scattering medium is zero and is unity respectively. The divergence of radiative heat flux has a critical role in both of the following i.e., radiation dominated process and combined mode of heat transfer. If there is no heat source/sink, a system stays in radiative equilibrium in case other modes of heat transfer are insignificant.
Under such conditions $\nabla \times q_r = 0$ where q_r is the radiative heat flux. With $I_b = \sigma T^4/\pi$ as the Planck's blackbody intensity and G as the incident radiation, in this case, in any control volume, emissions and absorptions are balanced. Since

$$\nabla \times q_r = 0$$

$$k_a (4\pi I_b - G) = 0 \quad (2a)$$

it follows that

$$I_b = \frac{G}{4\pi} \quad (2b)$$

The radiation equilibrium condition and isotopic scattering are assumed, Eq.(1) can be written as

$$\frac{dI}{ds} = \hat{s}.\nabla I = \beta\left(\frac{G}{4\pi} - I\right) \quad (3)$$

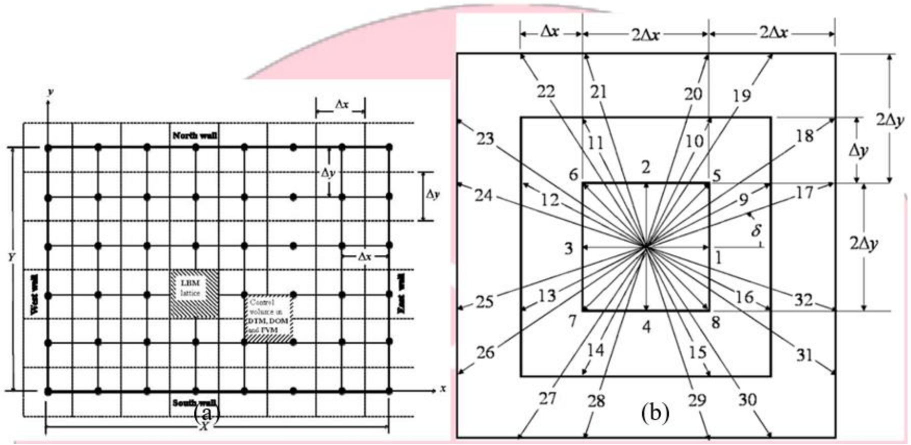


Fig. 1. (a) Arrangement of lattice and control volume in a 2-D rectangular geometry,
(b) Schematic of lattice D2Q8, D2Q16 and D2Q32.

Lattice Boltzmann Method (LBM) Formulation

LBM as proposed is employed to simulate radiative energy using particle distribution functions (PDFs) through which radiative energy is transmitted to the neighboring lattices only in some discrete directions. Consider the 2-D square enclosure having homogeneous, absorbing, emitting and scattering medium with diffusive and gray boundaries in Figure 1(a). The south wall is at temperature T_s as a source of radiation, whereas the other three walls are cold. We have assumed isotropy in the polar direction θ $(0 \le \theta \le \pi)$ and therefore we have considered angular dependence of intensity only in the azimuthal direction ϕ $(0 \le \phi \le 2\pi)$. Different types of lattices are used in the present LBM formulation as shown in Figure 2(a, b). For streaming the PDFs are used only in the finite discrete directions and isotropy is imposed in polar direction whereas for the calculation of heat fluxes and the incident radiation weights are employed to all intensities in the discrete directions which are spanned from 0 to 2π, see Figure 1(b). The discrete form of Eq. (2) is as follows:

$$\frac{dI_i}{ds} = \hat{s}.\nabla I_i = \beta\left(\frac{G}{4\pi} - I_i\right) \quad (4a)$$

Where I_i is the intensity in the discrete direction i and the transient form of equation [26] is:

$$\frac{1}{c}\frac{dI_i}{dt} + \hat{s}.\nabla I_i = \beta\left(\frac{G}{4\pi} - I_i\right) \quad (4b)$$

In the LBM formulation proposed by Asinari et al. [27] the azimuthal angle is discretized by introducing a finite number of discrete velocities ($\bar{e_i}$), lying on the lattice , whose magnitude is given by

$$e_i = \left|\vec{e_i}\right| = \sqrt{e_{xi} + e_{yi}} \quad (5)$$

Multiplying Eq. (3) by e_i with the assumption that $e_i = c$, we obtain the following equation:

$$\frac{dI_i}{dt} + \vec{e_i}.\nabla I_i = e_i\beta\left(\frac{G}{4\pi} - I_i\right); i = 1, \dots, N \quad (6)$$

where N is the total number of discrete directions. For the LBM scale the speed of light along each discrete direction of considered computational lattice is assumed to be fictitious i.e., $e_i = c$.

Although such an assumption leaves no room for a real transient description as shown in Eq. (4), it nevertheless provides quite a useful numerical technique for the solution of steady-state problems.

Applying now the forward Euler approximation to Eq. (5) we get the usual LBM formulation, namely:

$$I_i(\vec{r_n} + \vec{e_i}\Delta t, t + \Delta t) = I_i(\vec{r_n}, t) + \frac{\Delta t}{\tau_i}\left[I_i^{(eq)}(\vec{r_n}, t) - I_i(\vec{r_n}, t)\right], \quad (7)$$

where τ_i is the relaxation time which is defined as

$$\tau_i = \frac{1}{e_i \beta}, \quad (8)$$

$I_i^{(eq)}$ is the PDF distribution function which is computed from

$$I_i^{(eq)} = \sum_{i=1}^{N} I_i w_{gi}, \quad (9)$$

and w_{gi} is the weight in the discrete direction i, which can be computed from

$$w_{gi} = \left(\frac{1}{4\pi}\right)\int_0^{\pi} \sin\theta \, d\theta \int_{\phi-\frac{\Delta\phi}{2}}^{\phi+\frac{\Delta\phi}{2}} d\phi = \frac{\Delta\phi_i}{2\pi} \quad (10)$$

We have used the D2Q32 formulation in which the velocities of all PDFs I_i are not the same as in Figures 2(a) and 2(b).
The velocities of directions are given by:

$$e_{1,3} = (\pm 1, 0).U \qquad e_{2,4} = (0, \pm 1).U \qquad e_{5,6,7,8} = (\pm 1, \pm 1).U$$
$$e_{9,12} = (\pm 2, 1).U \qquad e_{13,16} = (\mp 2, -1).U$$
$$e_{10,11} = (\pm 1, 2).U \qquad e_{14,15} = (\mp 1, -2).U$$
$$e_{17,24} = (\pm 3, 1).U \qquad e_{25,32} = (\mp 3, -1).U$$
$$e_{18,23} = (\pm 3, 2).U \qquad e_{26,31} = (\mp 3, -2).U$$
$$e_{19,22} = (\pm 2, 3).U \qquad e_{27,30} = (\mp 2, -3).U$$
$$e_{20,21} = (\pm 1, 3).U \qquad e_{28,29} = (\pm 1, -3).U$$

The heat fluxes along x and y faces of enclosure are calculated as:

$$q_x = \sum_{i=1}^{N} I_i w_{xi} \quad \text{and} \quad q_y = \sum_{i=1}^{N} I_i w_{yi} \quad (11)$$

Where w_{xi} and w_{yi} are weights calculated as

$$w_{xi} = \int_0^{\pi} \sin^2\theta \, d\theta \int_{\phi-\frac{\Delta\phi}{2}}^{\phi+\frac{\Delta\phi}{2}} \cos\phi \, d\phi = \pi \cos\theta_i \sin\left(\frac{\Delta\phi_i}{2}\right), \quad (12)$$

$$w_{yi} = \int_0^{\pi} sin^2\,\theta\, d\theta \int_{\phi-\frac{\Delta\phi}{2}}^{\phi+\frac{\Delta\phi}{2}} sin\,\phi\, d\phi = \pi\, sin\,\theta_i\, sin(\frac{\Delta\phi_i}{2}). \quad (13)$$

For the calculation procedure the algorithm is split into collision and streaming steps i.e.,

$$I_i^*(\vec{r_n},t) = I_i(\vec{r_n},t) + \frac{\Delta t}{\tau_i}[I_i^{(eq)}(\vec{r_n},t) - I_i(\vec{r_n},t)]$$
$$I_i(\vec{r_n} + \vec{e_i}\Delta t, t+\Delta t) = I_i^*(\vec{r_n},t) \quad (14)(15)$$

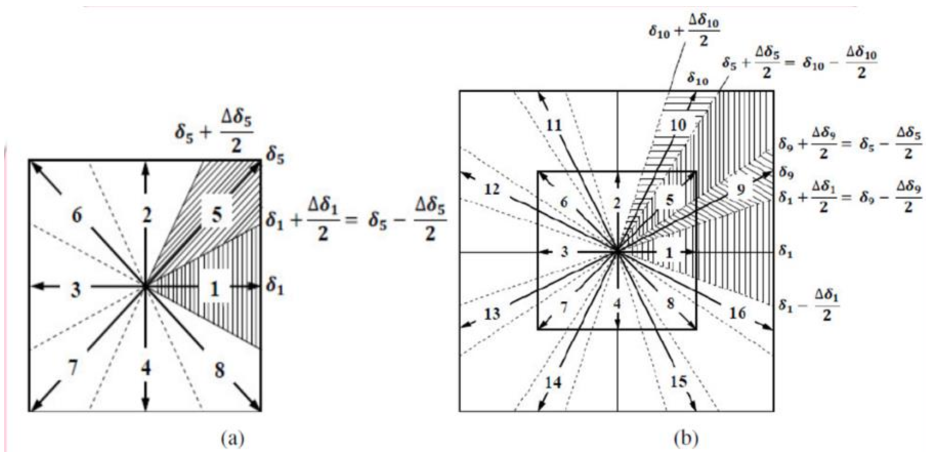


Fig. 2. Region of influence for the particle distribution function for (a) D2Q8, (b) D2Q16.

Finite Volume Method (FVM) Formulation

The discretization equation for radiative intensity can be obtained by integrating the RTE over typical control volume, a control angle For a typical control angle *l*, equation (1) can be written as

$$\frac{dI^l}{ds} = -\beta I^l + k_a I_b + \frac{\sigma_s}{4\pi}\sum_{l'=l}^{N} I^l p^{l'l}\,\Delta\Omega^{l'} = -(\beta - \frac{\sigma_s}{4\pi}p^{ll}\Delta\Omega^l)I^l + kI_b + \frac{\sigma_s}{4\pi}\sum_{\substack{l'=l \\ l'\neq l}}^{N} I^l p^{l'l}\,\Delta\Omega^{l'} \quad (16)$$

The linearized form of RTE equation is

$$\frac{dI^l}{ds} = -\beta^l{}_m I^l + S_m^l \quad (17)$$

where the modified extinction coefficient,

$$\beta_m^l = \beta - \frac{\sigma_s}{4\pi}p^{ll}\Delta\Omega^l \quad (18)$$

and modified source function,

$$S_m^l = kI_b + \frac{\sigma_s}{4\pi}\sum_{\substack{l'=l \\ l'\neq l}}^{N} I^l p^{l'l}\,\Delta\Omega^{l'} \tag{19}$$

After applying integration over a typical two-dimensional control volume and a control angle

$$a_P^l I_P^l = a_W^l I_W^l + a_E^l I_E^l + a_S^l I_S^l + a_N^l I_N^l + b^l, \tag{20}$$

where

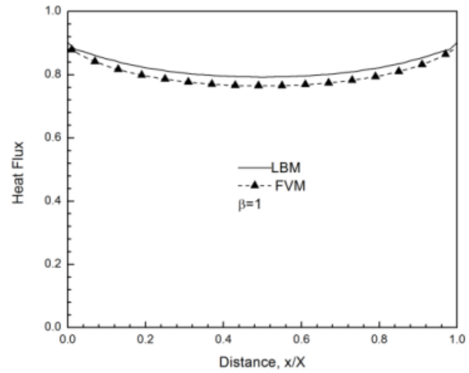
$a_E^l = \|-A_e D_{ce}^l\|,$ $a_W^l = \|-A_w D_{cw}^l\|,$

$a_N^l = \|-A_e D_{cn}^l\|,$ $a_S^l = \|-A_e D_{cs}^l\|,$

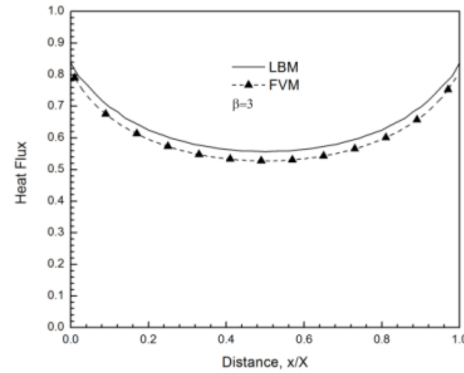
$a_p = \|A_e D_{ce}^l\| + \|A_w D_{cw}^l\| + \|A_e D_{cn}^l\| + \|A_e D_{cs}^l\| + \beta_{m,p}^l \Delta V_p \Delta\Omega^l,$ $b = S_{m,p}^l \Delta V_p \Delta\Omega^l.$

Results and Discussion

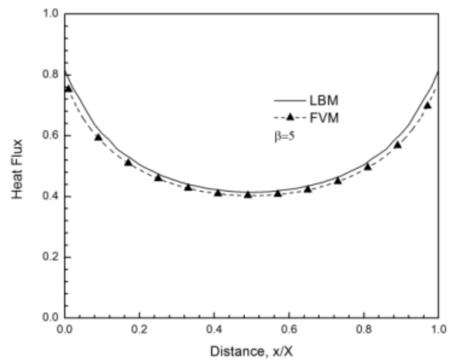
The above FVM and LBM formulations have been used for absorbing, emitting and scattering 2-D medium. As both methods are iterative computations have been started with guess values and results for steady state problems have been compared. To ensure the stability criteria $kn = \Delta x\beta \leq 0.05$ the 51×51 control volumes/lattices are used for the extinction coefficient $\beta \leq 5$, while 101×101 control volumes/lattices are used for $\beta > 5$. The comparisons have been made for D2Q32 for LBM and 8×16 for FVM azimuthal directions the convergence for both methods has been assumed when maximum change, between two successive iterations, at any point in incident radiation is less than $1\text{x}10^{-7}$. By applying LBM, dimensionless heat fluxes along hot wall for 6 values of extinction coefficients viz., β =1.0, 3.0, 5.0, 10.0, 15.0, 20.0 have been obtained in Figures (3a) – (3f) and compared with FVM reported by Mishra et al. [28]. The considered 2-D enclosure is at radiative equilibrium with south boundary being the source of radiation whereas all other boundaries are cold and black. The dimensionless heat flux along hot wall and emissivity for different values of extinction coefficient have been compared. It has been observed that participation of medium increases with increase in β hence causing a significant decrease in the net heat flux at south boundary. It is also pertinent to mention that increase in β raises the difference between values of heat flux at center of south wall and near the side wall. The reason behind the difference is that the center point of side walls receives more radiation from medium in comparison to the rest of points as medium becomes more participating with increase in β . For high values of β the results are in good agreement but little discrepancy is found for low values that can be overcome by increasing the number of directions in polar as well as azimuthal directions. In Figures 4(a) – 4(f) results of the dimensionless centerline ($x/x = 0.5, y$) emissive power distribution for wide range of extinction coefficient is compared. It has been observed that increase in β boosts up the temperature thus making the medium more diffusive. All results of LBM are in reasonable conformity with those of FVM whereas minor deviations can be overcome as mentioned above.



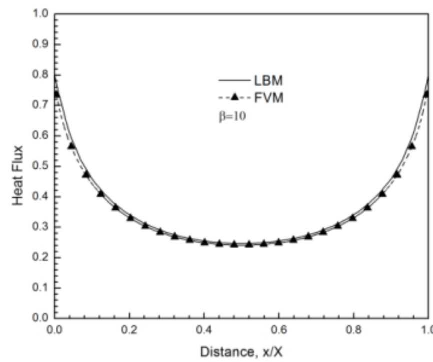
(a)



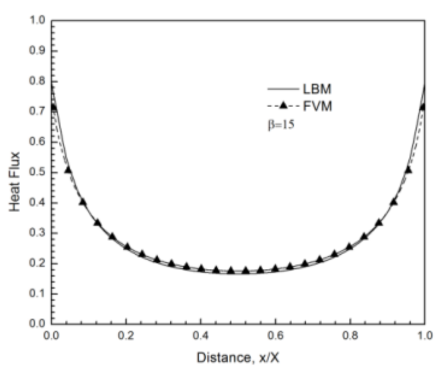
(b)



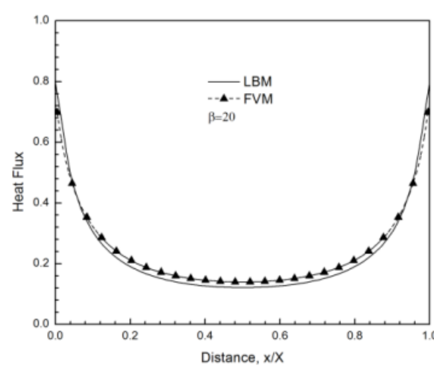
(c)



(d)



(e)



(f)

Fig. 3. Comparison of radiative heat flux along the bottom wall for different values of the extinction coefficient β.

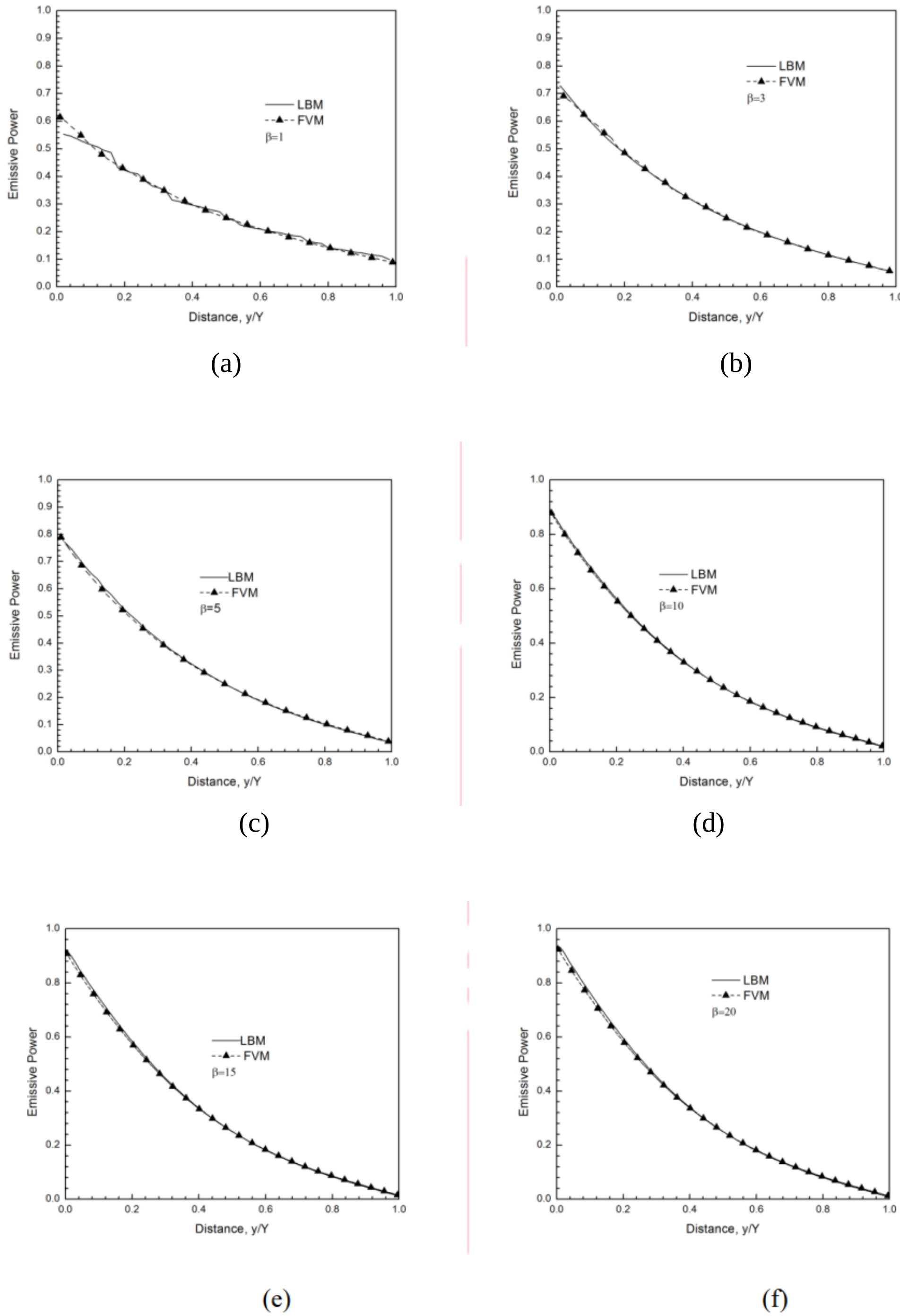
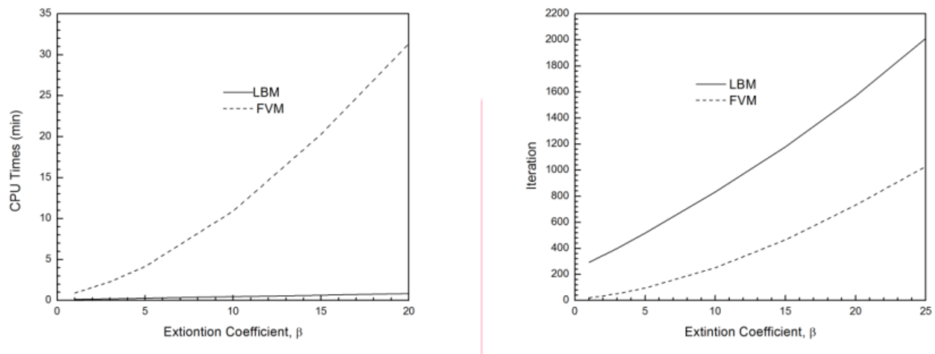


Fig. 4. Comparison of centerline emissive power for (a) β=1.0. (b) β=3.0.(c) β=5.0. (d) β=10.0. (e) β=15.0.

The comparison of both methods regarding CPU time and number of iterations has been made in Figures 5(a), 5(b) respectively. It has been noticed that number of iterations for convergence solution in LBM are quite higher than FVM whereas CPU time of the LBM is lesser. It implies that although LBM requires more iteration for convergence yet it consumes lesser time per iteration.



(a) (b)

Fig. 5. Comparison with change in the extinction coefficient of (a) CPU time and (b) number of Iterations.

2. Conclusion

The LBM is used to solve the radiative heat transfer in 2-D rectangular enclosure having absorbing emitting medium and scattering medium. The LBM formulation was tested against FVM for calculation of dimensional heat fluxes along hot wall and centerline emissive power with wide range of extinction coefficient. All of the LBM results are found to be in very good agreement with FVM, especially for high value of extinction coefficient. The comparison of both methods regarding CPU time and number of iterations is also made. It is found that for convergence of solution LBM takes a greater number of iteration than FVM whereas CPU time of the LBM is lesser. It is concluded that LBM is more efficient than FVM. The present work is implementation of the LBM for radiative transport problems a further study involves methodology to improve its accuracy and other type of problems.

Competing Interests The author declares no competing interests.

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